C4 June 2012

1.
$$f(x) = \frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

(a) Find the values of the constants A, B and C.

(b) (i) Hence find $\int f(x)\, dx$.

(ii) Find $\int_1^2 f(x)\, dx$, leaving your answer in the form $a + \ln b$,

where a and b are constants.

(6)

a) $1 = A(3x-1)^2 + B(x)(3x-1) + C(x)$

$x = 0 \Rightarrow 1 = A$

$x = \frac{1}{3} \Rightarrow 1 = \frac{1}{3}C \Rightarrow C = 3$

$x = 1 \Rightarrow 1 = 4A + 2B + C \Rightarrow 1 = 4 + 2B + 3 \quad B = -3$

b) $\int \frac{1}{x} + \frac{-3}{3x-1} + \frac{3}{(3x-1)^2}\, dx$

$(3x-1)^{-1}$
$\downarrow$
$-1(3x-1)^{-2} \times 3$

$\therefore \ln x - \ln(3x-1) - \frac{1}{3x-1} + c$

(ii) $\left[\ln\left(\frac{x}{3x-1}\right) - \frac{1}{3x-1} \right]_1^2$

$= \ln\frac{2}{5} - \frac{1}{5} - \ln\frac{1}{2} + \frac{1}{2} \qquad = \frac{3}{10} + \ln\frac{4}{5}$

2.

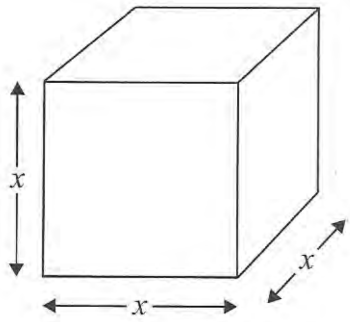

Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated.
At time t seconds, the length of each edge of the cube is x cm,
and the volume of the cube is V cm^3.

(a) Show that $\frac{dV}{dx} = 3x^2$ **(1)**

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^{3}s^{-1},

(b) find $\frac{dx}{dt}$, when $x = 8$ **(2)**

(c) find the rate of increase of the total surface area of the cube, in cm^{2}s^{-1}, when $x = 8$ **(3)**

a) $V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$ #

b) $\frac{dV}{dt} = 0.048$ $\qquad \frac{dx}{dV} \times \frac{dV}{dt} = \frac{0.048}{3x^2}$

$x = 8 \quad \frac{dx}{dt} = \frac{1}{4000}$

c) $A = 6x^2 \Rightarrow \frac{dA}{dx} = 12x$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} = \frac{12x}{4000} = \frac{3}{125}$

3.

$$f(x) = \frac{6}{\sqrt{(9-4x)}}, \quad |x| < \frac{9}{4}$$

(a) Find the binomial expansion of $f(x)$ in ascending powers of x, up to and the term in x^3. Give each coefficient in its simplest form.

Use your answer to part (a) to find the binomial expansion in ascending powers of x, up to and including the term in x^3, of

(b) $g(x) = \dfrac{6}{\sqrt{(9+4x)}}, \quad |x| < \dfrac{9}{4}$

(1)

(c) $h(x) = \dfrac{6}{\sqrt{(9-8x)}}, \quad |x| < \dfrac{9}{8}$

(2)

$$f(x) = 6(9-4x)^{-\frac{1}{2}} = 6 \times 9^{-\frac{1}{2}}\left(1 - \tfrac{4}{9}x\right)^{-\frac{1}{2}} = 2\left(1-\tfrac{4}{9}x\right)^{-\frac{1}{2}}$$

$$= 2\left[1 + \left(-\tfrac{1}{2}\right)\left(-\tfrac{4}{9}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\tfrac{4}{9}x\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(-\tfrac{4}{9}x\right)^3\right]$$

$$= 2 + \frac{4}{9}x + \frac{4}{27}x^2 + \frac{40}{729}x^3$$

b) $g(x) = 2 - \frac{4}{9}x + \frac{4}{27}x^2 - \frac{40}{729}x^3$

c) $h(x) = 2 + \frac{8}{9}x + \frac{16}{27}x^2 + \frac{320}{729}x^3$

4. Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}$$

$$\int y\,dy = \int 3\sec^2 x\,dx$$

$$\frac{1}{2}y^2 = 3\tan x + C$$

$$\left(\frac{\pi}{4}, 2\right) \quad 2 = 3\tan\frac{\pi}{4} + C \Rightarrow 2 = 3 + C \Rightarrow C = -1$$

$$\therefore \frac{1}{2}y^2 = 3\tan x - 1$$

$$y = \sqrt{6\tan x - 2}$$

5. The curve C has equation

$$16y^3 + 9x^2y - 54x = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(b) Find the coordinates of the points on C where $\frac{dy}{dx} = 0$.

(7)

$\frac{d}{dx}16y^3 + \frac{d}{dx}9x^2y - \frac{d}{dx}54x = 0$

$u = 9x^2 \quad v = y$

$u' = 18x \quad v' = \frac{dy}{dx}$

$\frac{dy}{dx}48y^2 + 9x^2\frac{dy}{dx} + 18xy - 54 = 0$

$(48y^2 + 9x^2)\frac{dy}{dx} = 54 - 18xy$

$\therefore \frac{dy}{dx} = \frac{54 - 18xy}{48y^2 + 9x^2} = \frac{18 - 6xy}{16y^2 + 3x^2}$

b) $\frac{dy}{dx} = 0 \Rightarrow 6xy = 18 \Rightarrow xy = 3 \Rightarrow y = \frac{3}{x}$

$16\left(\frac{3}{x}\right)^3 + 9x^{\cancel{2}}\left(\frac{3}{\cancel{x}}\right) - 54x = 0$

$\frac{432}{x^3} + 27x - 54x = 0$

$\Rightarrow \frac{432}{x^3} = 27x \Rightarrow x^4 = \frac{432}{27} = 16$

$\therefore x = \pm 2 \qquad (2, \frac{3}{2}) ; (-2, -\frac{3}{2})$

6.

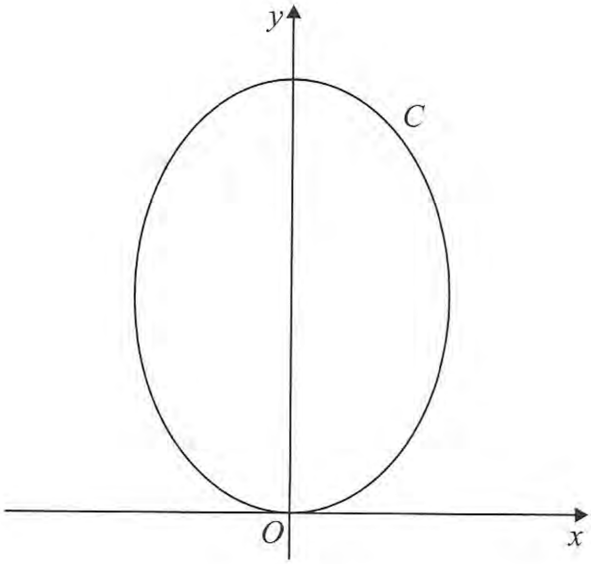

Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = (\sqrt{3})\sin 2t, \quad y = 4\cos^2 t, \quad 0 \leqslant t \leqslant \pi$$

(a) Show that $\frac{dy}{dx} = k(\sqrt{3})\tan 2t$, where k is a constant to be determined. **(5)**

(b) Find an equation of the tangent to C at the point where $t = \frac{\pi}{3}$.

Give your answer in the form $y = ax + b$, where a and b are constants. **(4)**

(c) Find a cartesian equation of C. **(3)**

$\frac{dy}{dt} = -8\cos t\sin t$ $\Rightarrow \frac{dy}{dx} = \frac{-4\sin 2t}{2\sqrt{3}\cos 2t}$

$\frac{dx}{dt} = 2\sqrt{3}\cos 2t$

$\frac{dy}{dx} = -\frac{2\sqrt{3}}{3}\tan 2t \quad (k = -\frac{2}{3})$

b) $t = \frac{\pi}{3}$ $\quad x = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$ $\quad y = 4\left(\frac{1}{2}\right)^2 = 1$

$m_t = -\frac{2\sqrt{3}}{3}(-\sqrt{3}) = 2 \Rightarrow y - 1 = 2\left(x - \frac{3}{2}\right)$

$y = 2x - 2$

c) $x = \sqrt{3} \times 2\sin t \cos t$

$x^2 = 12 \sin^2 t \cos^2 t$

$\therefore x^2 = 12\left(1-\frac{y}{4}\right)\left(\frac{y}{4}\right)$

$\frac{y}{4} = 1-$

$\sin^2 t = 1$

the following is not required.

$x^2 = 12\left(\frac{y}{4} - \frac{y^2}{16}\right) \Rightarrow x^2 = 3y - \frac{3}{4}y^2$

$\therefore 3y^2 - 12y + 4x^2 = 0$

$y^2 - 4y + \frac{4}{3}x^2 = 0$

$(y-2)^2 = 4 - \frac{4}{3}x^2$

$y - 2 = \pm\sqrt{4 - \frac{4}{3}x^2}$

$\therefore y = 2 \pm \sqrt{4 - \frac{4}{3}x^2}$

alt

$\sin 2t = \frac{x}{\sqrt{3}} \Rightarrow 2t = \arcsin\left(\frac{x}{\sqrt{3}}\right)$

$t = \frac{1}{2}\arcsin\left(\frac{x}{\sqrt{3}}\right)$

$\therefore y = 4\cos^2\left[\frac{1}{2}\arcsin\left(\frac{x}{\sqrt{3}}\right)\right]$

7.

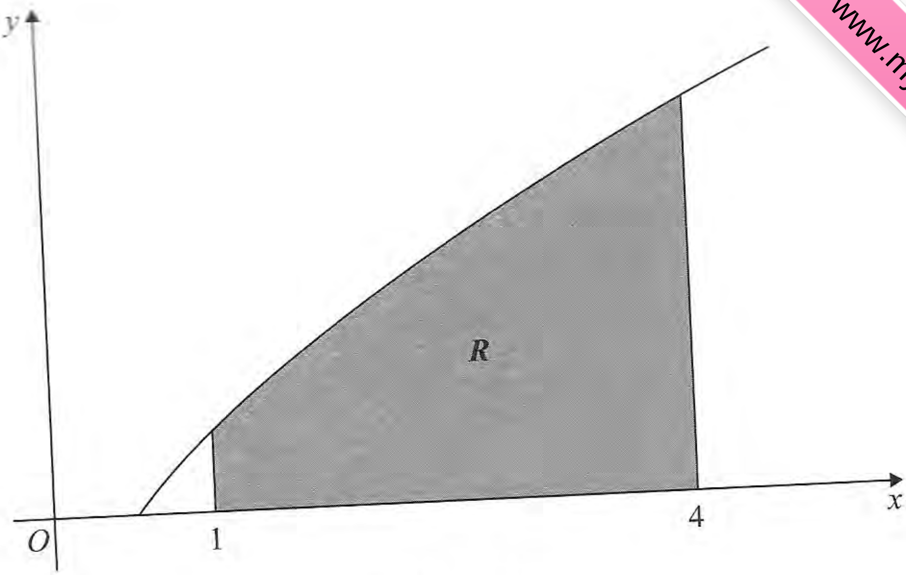

Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = x^{\frac{1}{2}} \ln 2x$.

The finite region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 4$

(a) Use the trapezium rule, with 3 strips of equal width, to find an estimate for the area of R, giving your answer to 2 decimal places. **(4)**

(b) Find $\int x^{\frac{1}{2}} \ln 2x \, dx$. **(4)**

(c) Hence find the exact area of R, giving your answer in the form $a \ln 2 + b$, where a and b are exact constants. **(3)**

x	1	2	3	4
y	$\ln 2$	$\sqrt{2}\ln 4$	$\sqrt{3}\ln 6$	$2\ln 8$

$$\text{Area} \simeq \frac{1}{2}(1)\left[\ln 2 + 2\ln 8 + 2\left(\sqrt{2}\ln 4 + \sqrt{3}\ln 6\right)\right]$$

$$\text{Area} \simeq \frac{14.98}{2} = 7.49$$

b) $\int x^{\frac{1}{2}} \ln 2x \, dx \qquad \int uv' = uv - \int u'v$

$u = \ln 2x \qquad v$

$u' = \frac{1}{x} \qquad v' = x$

$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \int \frac{\frac{2}{3} x^{\cancel{\frac{3}{2}}\,\frac{1}{2}}}{\cancel{x}} \, dx$

$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{4}{9}x^{\frac{3}{2}} + c$

c) $\int_1^4 x^{\frac{1}{2}} \ln 2x \, dx = \left[\frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{4}{9}x^{\frac{3}{2}} \right]_1^4$

$= \frac{16}{3}\ln 8 - \frac{32}{9} - \frac{2}{3}\ln 2 + \frac{4}{9}$

$= 16\ln 2 - \frac{2}{3}\ln 2 - \frac{28}{9}$

$= \frac{46}{3}\ln 2 - \frac{28}{9}$

8. Relative to a fixed origin O, the point A has position vector $(10\mathbf{i} + 2\mathbf{j}$
and the point B has position vector $(8\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$.

The line l passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$.

(

(b) Find a vector equation for the line l.

(2)

The point C has position vector $(3\mathbf{i} + 12\mathbf{j} + 3\mathbf{k})$.

The point P lies on l. Given that the vector $\overrightarrow{CP}$ is perpendicular to l,

(c) find the position vector of the point P.

(6)

a) $a = \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix}$ $b = \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix}$ $\overrightarrow{AB} = b - a = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10-2\lambda \\ 2+\lambda \\ 3+\lambda \end{pmatrix}$

c) $c = \begin{pmatrix} 3 \\ 12 \\ 3 \end{pmatrix}$

$\overrightarrow{CP} = \begin{pmatrix} 10-2\lambda \\ 2+\lambda \\ 3+\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 12 \\ 3 \end{pmatrix} = \begin{pmatrix} 7-2\lambda \\ -10+\lambda \\ \lambda \end{pmatrix}$

$\overrightarrow{CP}$ perp to L $\Rightarrow \begin{pmatrix} 7-2\lambda \\ -10+\lambda \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0$

$-14 + 4\lambda - 10 + \lambda + \lambda = 0 \Rightarrow 6\lambda = 24 \Rightarrow \lambda = 4$

$\therefore p = \begin{pmatrix} 10-8 \\ 2+4 \\ 3+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$